

Linearizability and fake Lax pair for a consistent around the cube nonlinear non-autonomous quad-graph equation.

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Abstract

We discuss the linearization of a non-autonomous nonlinear partial difference equation belonging to the Boll classification of quad-graph equations consistent around the cube. We show that its Lax pair is fake. We present its generalized symmetries which turn out to be non-autonomous and depending on an arbitrary function of the dependent variables defined in two lattice points. These generalized symmetries are differential difference equations which, in some case, admit peculiar Bäcklund transformations.

1 Introduction.

Symmetries and commuting flows have been in a way or in another at the base of integrability. Bäcklund transformations and nonlinear superposition rules [22, 29] paved the way to the discretization of integrable systems.

A first version of the integrability criteria denoted Consistency Around the Cube (*CAC*) can be found in the work of Doliwa and Santini [13].

In recent years *CAC* has been a source of many results in the classification of nonlinear difference equations. Its importance relies in the fact that provides Bäcklund transforms [7, 11, 27, 28] and as a consequence the existence of a zero curvature representation or Lax pairs, which, as it is well known [36] are associated to both linearizable and integrable equations.

The first attempt to carry out a classification of partial difference equations using the *CAC* condition has been presented in [1] assuming that the equations on all faces of the cube were the same form. The result is a class of discrete equations formed by systems living on quad-graphs, whose basic building blocks are equations on quadrilaterals of the type

$$A(x, x_1, x_2, x_{12}; \alpha_1, \alpha_2) = 0, \quad (1)$$

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where the four fields x, x_1, x_2 and $x_{12} \in \mathcal{C}$ are assigned to the four vertexes of a quadrilateral (later in (3) denoted as x_1, x_2, x_4 and x_4 or equivalently on a lattice of indices m and n as $x_{m,n}, x_{m+1,n}, x_{m,n+1}$ and $x_{m+1,n+1}$) and the parameters $\alpha_i \in \mathcal{C}, i = 1, 2$ to its edges ($\alpha_2 = \alpha_2(n), \alpha_1 = \alpha_1(m)$). In the notation defined

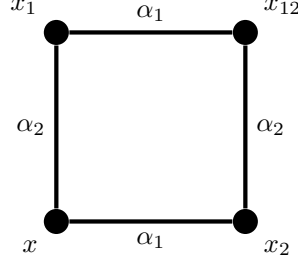


Figure 1: Quad-graph

above, if the quadrilateral is constructed designing two independent directions 1, 2 starting from an origin vertex on which the field x is assigned, then the subscript 1, 2 provide the field in the vertex shifted by α_1, α_2 along the direction 1, 2 from the origin, while 12 refers to the field in the remaining vertex of the quadrilateral. Moreover the function $A(x, x_1, x_2, x_{12}; \alpha_1, \alpha_2)$ is assumed to be affine linear in each argument (multilinearity) with coefficients depending on the two edge parameters and invariant under the discrete group D_4 of square symmetries

$$\begin{aligned} A(x, x_1, x_2, x_{12}; \alpha_1, \alpha_2) &= \varepsilon A(x, x_2, x_1, x_{12}; \alpha_2, \alpha_1) \\ &= \sigma A(x_1, x, x_{12}, x_2; \alpha_1, \alpha_2), \quad \varepsilon, \sigma = \pm 1. \end{aligned}$$

A last simplifying hypothesis is the so called *tetrahedron property* which amounts

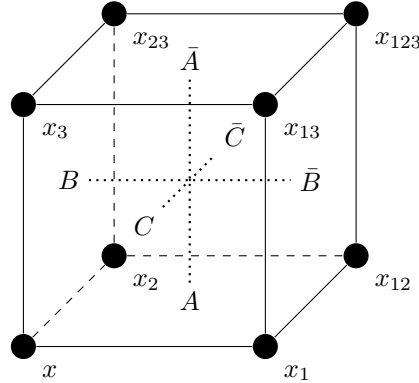


Figure 2: Equations on a Cube

to require that, starting from arbitrary initial values x, x_1, x_2 and x_3 , the function $x_{123} \doteq x_{123}(x, x_1, x_2, x_3; \alpha_1, \alpha_2, \alpha_3)$ does not depend on x . On a 3-D lattice of indices m, n and p , on the cube depicted in Fig. 2, $x = x_{m,n,p}, x_1 = x_{m+1,n,p}, x_2 = x_{m,n+1,p}$ and $x_3 = x_{m,n,p+1}$. Then $x_{123} = x_{m+1,n+1,p+1}$. The

lattice equation $A(x_{m,n,p}, x_{m+1,n,p}, x_{m,n+1,p}, x_{m+1,n+1,p}; \alpha_1(m), \alpha_2(n)) = 0$ is defined on the square of indices m and n and depends parametrically on the index p while $\bar{A} = 0$ is the same as $A = 0$ in the indices m and n but for a different value of $p, p + 1$. The equation $B = 0$ is defined on the square of indices m and p and depends parametrically on the index n while $\bar{B} = 0$ is defined on the square of indices n and p and depends parametrically on the index m . The final result of the classification is given by a set of two lists of equations, H and Q , for a total of seven consistent systems (up to a common Möbius transformation of the field variables and point transformations of edge parameters) are presented in [1]. So, in Fig. 2, on a 2-D lattice, \bar{A} represents a copy of the equation A (one of the seven equations H and Q) and on the faces B and \bar{B} we have a relation between a solution of A and one of \bar{A} , i.e. an auto-Bäcklund transformation for A . By going over to projective space the auto-Bäcklund transformation provide the Lax pair.

The same authors in [2] considered a more general perspective in the classification problem. They assumed that the faces of the consistency cube A, B, C and \bar{A}, \bar{B} and \bar{C} could carry a priori different quad-equations without assuming either the D_4 symmetry or the tetrahedron property. They considered six-tuples of (a priori different) quad-equations assigned to the faces of a 3D cube:

$$\begin{aligned} A(x, x_1, x_2, x_{12}; \alpha_1, \alpha_2) &= 0, & \bar{A}(x_3, x_{13}, x_{23}, x_{123}; \alpha_1, \alpha_2) &= 0, \\ B(x, x_2, x_3, x_{23}, \alpha_2, \alpha_3) &= 0, & \bar{B}(x_1, x_{12}, x_{13}, x_{123}, \alpha_2, \alpha_3) &= 0, \\ C(x, x_1, x_3, x_{13}; \alpha_1, \alpha_3) &= 0, & \bar{C}(x_2, x_{12}, x_{23}, x_{123}; \alpha_1, \alpha_3) &= 0, \end{aligned} \quad (2)$$

see Fig. 2. Such a six-tuple is *3D consistent* if, for arbitrary initial data x, x_1, x_2 and x_3 , the three values for x_{123} (calculated by using $\bar{A} = 0, \bar{B} = 0$ or $\bar{C} = 0$) coincide. As a result in [2] they reobtained the Q -type equations of [1] and showed some new examples of quad-equations of type H which turn out to be deformations of the previously one obtained [1]).

In [8–10], Boll classified all the consistent quad-equations possessing the tetrahedron property without any other additional assumption. All the results were summarized in a set of theorems, from Theorem 3.9 to Theorem 3.14 in [10], listing all the consistent six-tuples configurations up to $(\text{Möb})^8$, the group of independent Möbius transformations of the eight fields on the vertexes of the consistency cube. Defining for each equation (1) the accompanying biquadratics

$$A^{i,j} \equiv A^{i,j}(x_i, x_j) = A_{,x_m} A_{,x_n} - A A_{,x_m x_n}, \quad (3)$$

where $\{m, n\}$ is the complement of $\{i, j\}$ in $\{1, 2, 3, 4\}$, all the quad-equations fall into three disjoint families: Q -type (no degenerate biquadratic), H^4 -type (four biquadratics are degenerate) and H^6 -type (all of the six biquadratics are degenerate).

It's worth emphasizing that all the classification results holds locally, in the sense that everything is stated on a single quadrilateral cell or on a single cube. The non secondary problem of the embedding in a $2D/3D$ lattice of the single cell/single cube equations, so as to preserve $3D$ consistency, was already discussed in [2] introducing the concept of Black-White (BW) lattice. One way to solve this problem, is to embed (2) into a \mathbb{Z}^2 lattice with an elementary cell of dimension greater than one. In such a case the equation $Q = Q(x, x_1, x_2, x_{12}; \alpha_1, \alpha_2)$ can be extended to a lattice and the lattice equation will become integrable or linearizable. To do so, following [8], we reflect the

square with respect to the normal to its right and top sides and then complete a 2×2 lattice by reflecting again one of the obtained equation with respect to the other direction¹. Such procedure is graphically described in Figure 3, and at the level of the quad equation this correspond to construct the three equations obtained from $Q = Q(x, x_1, x_2, x_{12}; \alpha_1, \alpha_2)$ by flipping its fields:

$$Q = Q(x, x_1, x_2, x_{12}, \alpha_1, \alpha_2) = 0, \quad (4a)$$

$$|Q = Q(x_1, x, x_{12}, x_2, \alpha_1, \alpha_2) = 0, \quad (4b)$$

$$\underline{Q} = Q(x_2, x_{12}, x, x_1, \alpha_1, \alpha_2) = 0, \quad (4c)$$

$$|\underline{Q} = Q(x_{12}, x_2, x_1, x, \alpha_1, \alpha_2) = 0. \quad (4d)$$

By paving the whole \mathbb{Z}^2 with such equation we will get a partial difference equation, which we can in principle study with the known methods. Since *a priori* $Q \neq |Q \neq \underline{Q} \neq |\underline{Q}$ the obtained lattice will be a four color lattice, i.e. an extension of the BW lattice [21, 35].

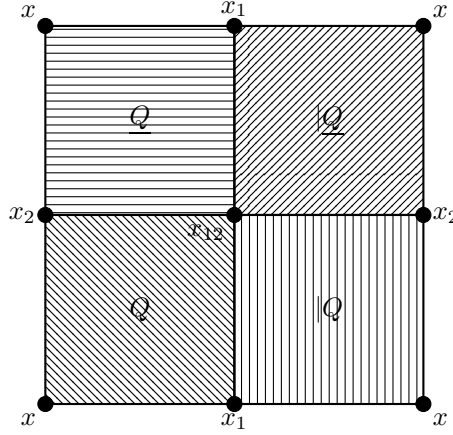


Figure 3: The “four colors” lattice

If the equation $Q = 0$ is invariant under the action of D_4 we have that:

$$Q = |Q = \underline{Q} = |\underline{Q} \quad (5)$$

implying that the elementary cell is actually of dimension one, and we fall into the case of the ABS classification. When

$$\begin{aligned} Q(x, x_1, x_2, x_{12}, \alpha_1, \alpha_2) &= \sigma Q(x, x_2, x_1, x_{12}, \alpha_2, \alpha_1) \\ &= \varepsilon Q(x_{12}, x_1, x_2, x, \alpha_2, \alpha_1), \end{aligned} \quad \sigma, \varepsilon \in \{ \pm 1 \} \quad (6)$$

i.e.

$$Q = |\underline{Q}, \quad \underline{Q} = |Q \quad (7)$$

the equation has *rhombic symmetry* while if

$$Q(x, x_1, x_2, x_{12}, \alpha_1, \alpha_2) = Q(x_1, x, x_{12}, x_2, \alpha_1, \alpha_2). \quad (8)$$

¹Let us note that, whatsoever side we reflect, the result of the last reflection is the same.

i.e.

$$Q = |Q, \quad \underline{Q} = |\underline{Q} \quad (9)$$

or

$$Q = \underline{Q}, \quad |Q = |\underline{Q} \quad (10)$$

we say that the equation has *trapezoidal symmetry*. A detailed study of all the lattices derived from the *rhombic* H^4 family, including the construction of the Lax pairs, Bäcklund transformations and infinite hierarchies of generalized symmetries, was presented in [35]. In [16] they studied all the equations of the Boll classification not already considered in the previous literature [1, 2, 35] and showed by using the algebraic entropy [33, 34] that they are linearizable and provided their explicit linearization.

A general procedure for the embedding is given in [8], [10]. Different embeddings in 3D consistent lattices resulting either in integrable or non integrable equations are discussed in [21] using an algebraic entropy analysis.

After the original work of Adler Bobenko and Suris there have been various attempts to simplify the requirements imposed on consistent quad-equations. In this way four non tetrahedral models, three of them with D_4 symmetry, were presented in [18, 19]. All of these models are linear equations or (more or less trivially) linearizable [30]. Other non tetrahedral, consistent systems of linear quadrilateral lattice equations were studied in [2, 5].

The existence of fake Lax pairs is well known but the phenomenon is not widely understood. The term Lax pair refers to a pair of linear equations (an overdetermined system) associated with a nonlinear integrable system through a compatibility condition. The most important property of a Lax pair is that it should prove the integrability of the associated nonlinear system and provide information about nontrivial solutions to nonlinear integrable system.

Fake Lax pairs are Lax pairs which tell us nothing about the integrability of the associated nonlinear system. Fake Lax pairs often appear very similar to their integrable counterparts and experts in the area of integrable systems continue to inadvertently publish fake Lax pairs that they believe are real (see [17] for examples). Moreover fake Lax pairs are usually associated to linearizable systems and to prove the fakeness of a Lax pair is a way to prove the linearizability of a system.

Lax pairs can be discrete or continuous, matrix or scalar, used for inverse scattering or isomonodromy, and fake Lax pairs reside in all of these categories. Although there are many references to fake Lax pairs in the literature, the most famous being [12], there are fewer articles that set out to explain what fake Lax pairs are and how to identify them. Various methods have been given to identify fake Lax pairs [17, 23–26, 31, 32].

In the following in Section 2 we focus on the simplest of the H^4 equations, the ${}_tH_1^\varepsilon$ equation, and present its Lax pair. In Section 3 we show its direct linearizability and that the obtained Lax pair is effectively fake. In Section 4 we present its generalized symmetries which turn out to be differential difference equations which have, by construction Bäcklund transformations. Section 5 is devoted to some final conclusions.

2 The ${}_tH_1^\varepsilon$ equation.

Before defining the equation ${}_tH_1^\varepsilon$ we need to introduce some notation:

$$\mathcal{F}_p^{(\pm)} \doteq \frac{1 \pm (-1)^p}{2}, \quad \alpha_1 \doteq \alpha_1(m), \quad \alpha_2 \doteq \alpha_2(n), \quad \alpha_3 \doteq \alpha_3(p), \quad (11)$$

where α_1, α_2 are constants which appear in the equation and might depend on the lattice variable and play the role of the lattice spacing when carrying out the continuous limit. α_3 will appear in the Bäcklund transformations and will play the role of the spectral parameter.

$$\begin{aligned} \psi_{m,n+1} &= \ell_{m,n} \cdot L_{m,n} \psi_{m,n}, \quad \psi_{m+1,n} = \mathfrak{m}_{m,n} \cdot M_{m,n} \psi_{m,n}, \quad \tau_{m,n} \doteq \frac{\mathfrak{m}_{m,n} \ell_{m+1,n}}{\ell_{m,n} \mathfrak{m}_{m,n+1}}, \\ \tau_{m,n} \cdot L_{m+1,n} \cdot M_{m,n} - M_{m,n+1} \cdot L_{m,n} &= 0, \\ \ell_{m,n} &\doteq \ell(m, n, x_{m,n}, x_{m,n+1}; \alpha_3, \alpha_1, \varepsilon), \quad L_{m,n} \doteq L(m, n, x_{m,n}, x_{m,n+1}; \alpha_3, \alpha_1, \varepsilon), \\ \mathfrak{m}_{m,n} &\doteq \mathfrak{m}(m, n, x_{m,n}, x_{m+1,n}; \alpha_2, \alpha_3, \varepsilon), \quad M_{m,n} \doteq M(m, n, x_{m,n}, x_{m+1,n}; \alpha_2, \alpha_3, \varepsilon). \end{aligned} \quad (12)$$

The elements L and M represent the Lax pair on the m and n lattice and (12) is the corresponding Lax equation on the lattice. The related normalization coefficients are denoted ℓ and \mathfrak{m} and τ is their ratio which appear in the Lax equation. α_3 place the role of the spectral parameter.

Then the ${}_tH_1^\varepsilon$ equation reads:

$$(x_{m,n} - x_{m+1,n})(x_{m,n+1} - x_{m+1,n+1}) - \varepsilon^2 \alpha_2 \left(\mathcal{F}_n^{(+)} x_{m,n+1} x_{m+1,n+1} + \mathcal{F}_n^{(-)} x_{m,n} x_{m+1,n} \right) - \alpha_2 = 0. \quad (13)$$

(13) it is obtained from the cell equation

$$(x - x_2)(x_3 - x_{23}) - \alpha_2(1 + \varepsilon^2 x_3 x_{23}) = 0, \quad (14)$$

by the procedure schematized in the introduction and described in detail in [16].

Its Lax pair and normalization coefficients are:

$$L_{m,n} = \begin{pmatrix} x_{m,n+1} & -x_{m,n} x_{m,n+1} + \alpha_3 \\ 1 & -x_{m,n} \end{pmatrix} - \varepsilon^2 \alpha_3 \begin{pmatrix} -\mathcal{F}_n^{(-)} x_{m,n} & 0 \\ 0 & \mathcal{F}_n^{(+)} x_{m,n+1} \end{pmatrix}, \quad (15)$$

$$M_{m,n} = \begin{pmatrix} \alpha_3(x_{m,n} - x_{m+1,n}) + \alpha_2 x_{m+1,n} & -\alpha_2 x_{m,n} x_{m+1,n} \\ \alpha_2 & \alpha_1(x_{m,n} - x_{m+1,n}) - \alpha_2 x_{m,n} \end{pmatrix} - \quad (16)$$

$$\begin{aligned} -\varepsilon^2 \alpha_3 \alpha_2 (\alpha_3 - \alpha_2) \mathcal{F}_n^{(+)} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau_{m,n} &= \frac{\alpha_2 \left(1 + \varepsilon^2 \mathcal{F}_n^{(-)} x_{m,n}^2 + \varepsilon^2 \mathcal{F}_n^{(+)} x_{m,n+1}^2 \right)}{(x_{m,n} - x_{m+1,n})^2 + \varepsilon^2 \alpha_2^2 \mathcal{F}_n^{(+)}}, \\ \ell_{m,n} &= \frac{1}{1 \mp i\varepsilon \left(\mathcal{F}_n^{(-)} x_{m,n} - \mathcal{F}_n^{(+)} x_{m,n+1} \right)}, \quad \mathfrak{m} = \frac{1}{x_{m,n} - x_{m+1,n} \pm i\varepsilon \alpha_2 \mathcal{H}_n^{(+)}}, \end{aligned} \quad (17)$$

where, with a completely non trivial calculation, one has been able to rewrite the connection formulas (17) without using the standard square root terms. The equations associated to the other sides of the cube are:

$$\alpha_2(x_{m,p} - x_{m,p+1})(x_{m+1,p} - x_{m+1,p+1}) - \alpha_3(x_{m,p} - x_{m+1,p})(x_{m,p+1} - x_{m+1,p+1}) + \quad (18)$$

$$\begin{aligned}
& +\varepsilon^2\alpha_3\alpha_2(\alpha_3-\alpha_2)\mathcal{F}_q^{(+)}=0, \\
& (x_{n,q}-x_{n,q+1})(x_{n+1,q}-x_{n+1,q+1})-\varepsilon^2\alpha_3\left(\mathcal{F}_n^{(+)}x_{n+1,q}x_{n+1,q+1}+\mathcal{F}_n^{(-)}x_{n,q}x_{n,q+1}\right)-\alpha_3=0.
\end{aligned}$$

Eqs. (18) are the Bäcklund transformations for the ${}_tH_1^\varepsilon$ equation when the lattice variables $p+1$ and $q+1$ are interpreted as indices indicating a new solution. By going over to projective space (18) provide the Lax pair (16).

3 Linearization and fake Lax pairs.

Let us now analyze equation (13). In (13) it is always possible to suppose $\alpha_2 \neq 0$, otherwise the equation degenerates into $(x_{m,n}-x_{m+1,n})(x_{m,n+1}-x_{m+1,n+1})=0$ whose solution is trivial on the lattice. Let us define $x_{m,2k} \doteq w_{m,k}$, $x_{m,2k+1} \doteq z_{m,k}$; then we have the following system of two coupled autonomous difference equations

$$(w_{m,k}-w_{m+1,k})(z_{m,k}-z_{m+1,k})-\varepsilon^2\alpha_2z_{m,k}z_{m+1,k}-\alpha_2=0, \quad (19a)$$

$$(w_{m,k+1}-w_{m+1,k+1})(z_{m,k}-z_{m+1,k})-\varepsilon^2\alpha_2z_{m,k}z_{m+1,k}-\alpha_2=0. \quad (19b)$$

Subtracting (19b) to (19a), we obtain

$$(w_{m,k}-w_{m+1,k}-w_{m,k+1}+w_{m+1,k+1})(z_{m,k}-z_{m+1,k})=0. \quad (20)$$

At this point the solution of the system bifurcates:

- **Case 1:** if $z_{m,k} = f_k$, where f_k is a generic function of its argument, equation (20) is satisfied and from (19a) or (19b) we have that $\varepsilon \neq 0$ and, solving for f_k , one gets

$$f_k = \pm \frac{i}{\varepsilon}. \quad (21)$$

- **Case 2:** if $z_{m,k} \neq f_k$, with f_k given in (21), one has $w_{m,k} = g_m + h_k$, where g_m and h_k are arbitrary functions of their argument. Hence (19b) and (19a) reduce to

$$\varepsilon^2z_{m,k}z_{m+1,k} + \kappa_m(z_{m,k}-z_{m+1,k}) + 1 = 0, \quad \kappa_m \doteq \frac{g_{m+1}-g_m}{\alpha_2}, \quad (22)$$

so that two sub-cases emerge:

- **Sub-case 2.1:** if $\varepsilon = 0$, (22) implies $\kappa_m \neq 0$, so that, solving,

$$z_{m+1,k} - z_{m,k} = \frac{1}{\kappa_m}, \quad (23)$$

we get

$$z_{m,k} = j_k + \sum_{l=m_0}^{m-1} \frac{1}{\kappa_l}, \quad m \geq m_0 + 1, \quad (24a)$$

$$z_{m,k} = j_k - \sum_{l=m}^{m_0-1} \frac{1}{\kappa_l}, \quad m \leq m_0 - 1, \quad (24b)$$

where $j_k = z_{m_0,k}$ is a generic integration function of its argument.

- **Sub-case 2.2:** if $\varepsilon \neq 0$, (22) is a discrete Riccati equation which can be linearized by the Möbius transformation $z_{m,k} \doteq \frac{i y_{m,k} - 1}{\varepsilon y_{m,k} + 1}$ to

$$(i\kappa_m - \varepsilon) y_{m+1,k} = (i\kappa_m + \varepsilon) y_{m,k}, \quad (25)$$

which, as $\kappa_m \neq \pm i\varepsilon$ because otherwise $y_{m,k} = 0$ and $z_{m,k} = -i/\varepsilon$. Eq. (25) implies

$$y_{m,k} = j_k \prod_{l=m_0}^{m-1} \frac{i\kappa_l + \varepsilon}{i\kappa_l - \varepsilon}, \quad m \geq m_0 + 1, \quad (26a)$$

$$y_{m,k} = j_k \prod_{l=m}^{m_0-1} \frac{i\kappa_l - \varepsilon}{i\kappa_l + \varepsilon}, \quad m \leq m_0 - 1, \quad (26b)$$

where $j_k = y_{m_0,k}$ is another arbitrary integration function of its argument.

In conclusion we have always completely integrated the original system.

Let us note that in the case $\varepsilon = 0$ when (13) becomes

$$(x_{m,n} - x_{m+1,n})(x_{m,n+1} - x_{m+1,n+1}) - \alpha_2 = 0, \quad (27)$$

the contact Möbius-type transformation

$$x_{m,n} - x_{m+1,n} = \sqrt{\alpha_2} \frac{2w_{m,n} + 1 - \alpha_2}{2w_{m,n} + 1 + \alpha_2}, \quad (28)$$

brings (13) into the following first order linear equation:

$$w_{m,n+1} + w_{m,n} + 1 = 0, \quad (29)$$

whose solution is:

$$x_{m,n} = x_{0,n} - \sqrt{\alpha_2} \sum_{l=1}^m \frac{2w_{l,0}(-1)^n + (-1)^n - \alpha_2}{2w_{l,0}(-1)^n + (-1)^n + \alpha_2}. \quad (30)$$

Here $x_{0,n}$ and $w_{m,0}$ are two arbitrary integration functions.

An other linearizing transformation is given by

$$x_{m,n} - x_{m+1,n} = \sqrt{\alpha_2} e_{m,n}^z, \quad (31)$$

bringing (13) into the following first order linear equation:

$$z_{m,n+1} + z_{m,n} = 2 * i\pi\kappa, \quad (32)$$

where κ is an arbitrary entire paramether.

The linearization can also be achieved using the Lax pair (15-17). Introducing the fields $\rho_{m,k} \doteq \tilde{\psi}_{m,2k}^{(1)}$, $\sigma_{m,k} \doteq \tilde{\psi}_{m,2k+1}^{(1)}$, $\theta_{m,k} \doteq \tilde{\psi}_{m,2k}^{(2)}$, $\chi_{m,k} \doteq \tilde{\psi}_{m,2k+1}^{(2)}$, the Lax pair, choosing the upper sign, can be rewritten as

$$\begin{aligned} \phi_{m,k+1} &= \mathcal{L}_{m,k} \phi_{m,k}, \quad \phi_{m+1,k} = \mathcal{M}_{m,k} \phi_{m,k}, \quad \phi_{m,k} \doteq \begin{pmatrix} \rho_{m,k} \\ \theta_{m,k} \end{pmatrix}, \\ \mathcal{L}_{m,k} &= -\alpha_3 \begin{pmatrix} 1 & w_{m,k+1} - w_{m,k} \\ 0 & 1 \end{pmatrix}, \end{aligned} \quad (33a)$$

$$\begin{aligned} \mathcal{M}_{m,k} &= \frac{1}{w_{m,k} - w_{m+1,k} + i\varepsilon\alpha_2} \begin{pmatrix} \alpha_3(w_{m,k} - w_{m+1,k}) + \alpha_2 w_{m+1,k} & -\alpha_2 w_{m,k} w_{m+1,k} - \varepsilon^2 \alpha_3 \alpha_2 (\alpha_3 - \alpha_2) \\ \alpha_2 & \alpha_1(w_{m,k} - w_{m+1,k}) - \alpha_2 w_{m,k} \end{pmatrix} \\ \begin{pmatrix} \sigma_{m,k} \\ \chi_{m,k} \end{pmatrix} &= \frac{1}{i - \varepsilon z_{m,k}} \begin{pmatrix} z_{m,k} & \alpha_3 - w_{m,k} z_{q,k} \\ 1 & -w_{m,k} - \varepsilon^2 \alpha_3 z_{m,k} \end{pmatrix} \phi_{m,k}, \end{aligned} \quad (33c)$$

$$(33d)$$

$$\alpha_3 \mathcal{E}_m \mathcal{N}_{q,k} \phi_{q,k} = 0, \quad (33e)$$

$$\mathcal{N}_{m,k} \doteq \begin{pmatrix} i\varepsilon(z_{m,k} - z_{m+1,k}) & i\varepsilon(w_{m,k} z_{m+1,k} - w_{m+1,k} z_{m,k}) + \varepsilon^2 \alpha_3 (z_{m,k} - z_{m+1,k}) - \varepsilon^2 \alpha_2 z_{m,k} + w_{m,k} - w_{m+1,k} + i\varepsilon\alpha_2 \\ z_{m,k} - z_{m+1,k} & w_{m+1,k} z_{m+1,k} - w_{m,k} z_{m,k} - i\varepsilon\alpha_3 (z_{m,k} - z_{m+1,k}) - i\varepsilon\alpha_2 z_{m+1,k} - i\varepsilon(w_{m,k} - w_{m+1,k} + i\varepsilon\alpha_2) z_{m,k} \end{pmatrix}$$

with $w_{m,k} - w_{m+1,k} + i\varepsilon\alpha_2 \neq 0$, $z_{m,k} - z_{m+1,k} \neq 0$ and \mathcal{E}_m is the left hand side of the equation (19a). In deriving (33a) and (33e) we have used the relation (33c) and its difference consequences. The compatibility between (33a) and (33b) implies $w_{m,k} - w_{m+1,k} - w_{m,k+1} + w_{m+1,k+1} = 0$, while (33e) implies $\mathcal{E}_m = 0$, which is (19a). Analogously one could have written any relation in terms of the vector $\begin{pmatrix} \sigma_{m,k} \\ \chi_{m,k} \end{pmatrix}$. In this case the compatibility of the so obtained Lax pair, after an integration, would imply (22).

Let us show that the Lax pair (33a, 33b) is fake. Under the gauge transformation

$$\phi_{m,k} = \mathcal{G}_{m,k} \phi'_{m,k}, \quad \mathcal{G}_{m,k} \doteq (-\alpha_3^k) \alpha_1^m \begin{pmatrix} 1 & w_{m,k} \\ 0 & 1 \end{pmatrix},$$

the Lax pair (33a, 33b) becomes

$$\mathcal{L}'_{m,k} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (34)$$

$$\mathcal{M}'_{m,k} = \frac{1}{w_{m,k} - w_{m+1,k} + i\varepsilon\alpha_2} \begin{pmatrix} w_{m,k} - w_{m+1,k} & (w_{m,k} - w_{m+1,k})^2 - \varepsilon^2 \alpha_2 (\alpha_3 - \alpha_2) \\ \frac{\alpha_2}{\alpha_3} & w_{m,k} - w_{m+1,k} \end{pmatrix} \quad (35)$$

In (34, 35) $\mathcal{L}'_{m,k}$ is independent both on the spectral parameter α_3 and on any field. So it is a useless matrix for solving any spectral problem.

4 Generalized Symmetries and their Bäcklund transformations.

In this Section we construct the three point generalized symmetries written as

$$x_{m,n;\lambda} = g_{m,n}(s, t, u, v, \bar{u}, \bar{v}, x_{m,n}; \alpha_2, \varepsilon), \quad s \doteq \frac{x_{m+1,n} - x_{m,n}}{1 + \varepsilon^2 x_{m+1,n} x_{m,n}},$$

$$t \doteq \frac{x_{m,n} - x_{m-1,n}}{1 + \varepsilon^2 x_{m-1,n} x_{m,n}}, \quad u \doteq x_{m+1,n} - x_{m,n}, \quad v \doteq x_{m,n} - x_{m-1,n}$$

$$\bar{u} \doteq x_{m,n+1} - x_{m,n}, \quad \bar{v} \doteq x_{m,n} - x_{m,n-1}$$

associated to the equation (13) in the m and n direction and then discuss the equation (13) as a Bäcklund transformation for the obtained differential difference equations.

Three-points generalized symmetries along direction m .

- **Case $\varepsilon \neq 0$:**

$$g_{m,n}^{(\varepsilon)} = \mathcal{F}_n^{(+)} \left\{ \frac{\alpha_2 (v^2 + \varepsilon^2 \alpha_2^2)}{(u-v)(u+v)} B\left(\frac{\alpha_2}{u}, m\right) - \frac{\alpha_2 (u^2 + \varepsilon^2 \alpha_2^2)}{(u-v)(u+v)} B\left(\frac{\alpha_2}{v}, m-1\right) + \right. \quad (36)$$

$$+ \left[x_{m,n} - \frac{(u^2 + \varepsilon^2 \alpha_2^2) v}{(u-v)(u+v)} \right] \omega + \gamma_n \Big\} + \mathcal{F}_n^{(-)} \left[\frac{s^2 t^2}{(s-t)(s+t)} (B(s, m) - B(t, m-1)) - \right.$$

$$\left. - \frac{s^2 t}{(s-t)(s+t)} \omega + \delta_n \right] (1 + \varepsilon^2 x_{m,n}^2),$$

where $B(y, m)$, γ_n and δ_n are generic functions of their arguments and ω is an arbitrary parameter. Let us note that any function and parameter may eventually depend on α_2 and ε . It is possible to demonstrate that, as long as $\varepsilon \neq 0$, no n -independent reduction of the above symmetry exists when $x_{m,n} \doteq x_m$;

- **Case $\varepsilon = 0$:**

In this case the equations are simpler and we get:

$$g_{m,n}^{(0)} = \frac{u^2 v^2}{(u-v)(u+v)} \left[\mathcal{F}_n^{(+)} \left(\frac{B(\frac{\alpha_2}{u}, m)}{u^2} - \frac{B(\frac{\alpha_2}{v}, m-1)}{v^2} \right) \alpha_2 + \right. \quad (37)$$

$$+ \mathcal{F}_n^{(-)} (B(u, m) - B(v, m-1)) \Big] + \left[\frac{x_{m,n}}{2} - \frac{u^2 v}{(u-v)(u+v)} \right] \omega + (-1)^n \sigma x_{m,n} + f_n,$$

where $B(y, m)$ and f_n are generic functions of their arguments and ω and σ are arbitrary parameters. As before any function and parameter may eventually depend on α_2 . It is easy to show that, taking $f_n = \frac{1+(-1)^n}{2} \gamma_n + \frac{1-(-1)^n}{2} \delta_n$, the $\varepsilon \rightarrow 0$ limit of the symmetry (36) coincides with the restriction $\sigma = \omega/2$ of the symmetry (37). When $\varepsilon = 0$ our equation becomes n -independent. Hence it is natural to ask if the family of symmetries (37) contains a n -independent reduction. This reduction exists and is given by

$$g_{m,n}^{(0)} = \frac{u^2 v^2}{(u-v)(u+v)} \left(\frac{B(u, m; \alpha_2)}{u} - \frac{B(v, m-1; \alpha_2)}{v} \right) + \left[\frac{x_{m,n}}{2} - \frac{u^2 v}{(u-v)(u+v)} \right] \omega + \chi, \quad (38)$$

where ω and χ are arbitrary parameters (eventually depending on α_2) and $B(y, m; \alpha_2)$ is a function of its arguments satisfying the following functional equation

$$B\left(\frac{\alpha_2}{y}, m; \alpha_2\right) = B(y, m; \alpha_2),$$

whose general solution reads

$$B(y, m; \alpha_2) = \begin{cases} G(y, m; \alpha_2), & y \in \mathcal{B}(\alpha_2) \subset \Omega(\alpha_2), \\ G\left(\frac{\alpha_2}{y}, m; \alpha_2\right), & \frac{\alpha_2}{y} \in \mathcal{B}(\alpha_2), \end{cases}$$

where $G(y, m; \alpha_2)$ is a generic function of its arguments, $\Omega(\alpha_2)$ represents the following subset of the complex plane

$$\Omega(\alpha_2) \doteq \left\{ z \in \mathcal{C} : |z| > \sqrt{|\alpha_2|} \cup z = \sqrt{|\alpha_2|} e^{i\theta_z} : \frac{\theta_{\alpha_2}}{2} \leq \theta_z \leq \frac{\theta_{\alpha_2}}{2} + \pi \right\}$$

and $\mathcal{B}(\alpha_2)$ is a generic subset of $\Omega(\alpha_2)$. In particular, if $B(y, m; \alpha_2)$ is an analytic function in the annulus centered at the origin and of radii $r_1(\alpha_2)$ and $r_2(\alpha_2)$ so that $r_1(\alpha_2) < \sqrt{|\alpha_2|} < r_2(\alpha_2)$, it is possible to show that

$$B(y, m; \alpha_2) = U\left(\frac{\alpha_2}{y} + y, m; \alpha_2\right),$$

$U(y, m; \alpha_2)$ being an analytic function of y in some domain of the complex plane but otherwise generic in all its arguments. For example, if in (38) we set $\omega = 0$ and we choose $U(y, m; \alpha_2) = -1$, we obtain the symmetry

$$\dot{x}_{m,n} = \frac{(x_{m+1,n} - x_{m,n})(x_{m,n} - x_{m-1,n})}{x_{m+1,n} - x_{m-1,n}} + \chi,$$

belonging to the list of Volterra-type integrable differential difference equations, cfr. [36], page. 597 (after the translation $x_{m,n} \doteq u_{m,n} + t\chi$).

Three-points generalized symmetries along direction n .

We have the following symmetries in the n direction:

$$\begin{aligned} g_{m,n}^{(\varepsilon)} &= \mathcal{F}_n^{(+)} \left(B_n \left(\frac{(\bar{u} + \bar{v})}{1 + \varepsilon^2 x_{m,n+1} x_{m,n-1}} \right) + \kappa_n \right) \\ &+ \mathcal{F}_n^{(-)} (1 + \varepsilon^2 x_{m,n}^2) (C_n(\bar{u} + \bar{v}) + \lambda_m), \end{aligned} \quad (39)$$

where $B_n(y)$ and $C_n(y)$ can be arbitrary functions of their argument and of the lattice variable n . κ_m and λ_m are arbitrary functions of the lattice variable m . In the case $\varepsilon = 0$ we have:

$$g_{m,n}^{(0)} = D_n(\bar{u} + \bar{v}) + (-1)^n \sigma x_{m,n} + \theta_n, \quad (40)$$

with $D_n(y)$ and θ_n being arbitrary functions of their argument and σ an arbitrary parameter. It is straightforward to show that, taking $D_n(y) = \mathcal{F}_n^{(+)} B_n(y) + \mathcal{F}_n^{(-)} C_n(y)$ and $\theta_n = \mathcal{F}_n^{(+)} \kappa_n + \mathcal{F}_n^{(-)} \lambda_n$, the $\varepsilon \rightarrow 0$ limit of the symmetry (39) coincide with the restriction $\sigma = 0$ of the symmetry (40).

4.1 Differential difference equations and Bäcklund transformations.

We now interpret (36), when $\varepsilon \neq 0$, as a differential difference equation in m . As (36) depends also on n we can do so only in the case that the dependence

is through $\mathcal{F}_n^{(+)}$ and $\mathcal{F}_n^{(-)}$ as in this case we can overcome it by going over to fields depending on even or odd values of n . In this case we must set γ_n and δ_n to zero and define the new fields w_m and z_m as in Section 3. So (36) becomes the following system of equations:

$$w_{m;\lambda}^{(\varepsilon)} = \left\{ \frac{\alpha_2 (\tilde{v}^2 + \varepsilon^2 \alpha_2^2)}{(\tilde{u} - \tilde{v})(\tilde{u} + \tilde{v})} B\left(\frac{\alpha_2}{\tilde{u}}, m\right) - \frac{\alpha_2 (\tilde{u}^2 + \varepsilon^2 \alpha_2^2)}{(\tilde{u} - \tilde{v})(\tilde{u} + \tilde{v})} B\left(\frac{\alpha_2}{\tilde{v}}, m-1\right) + \left[w_m - \frac{(\tilde{u}^2 + \varepsilon^2 \alpha_2^2) \tilde{v}}{(\tilde{u} - \tilde{v})(\tilde{u} + \tilde{v})} \right] \alpha \right\}, \quad (41)$$

$$z_{m;\lambda}^{(\varepsilon)} = \left[\frac{\tilde{s}^2 \tilde{t}^2}{(\tilde{s} - \tilde{t})(\tilde{s} + \tilde{t})} (B(\tilde{s}, m) - B(\tilde{t}, m-1)) - \frac{\tilde{s}^2 \tilde{t}}{(\tilde{s} - \tilde{t})(\tilde{s} + \tilde{t})} \omega \right] (1 + \varepsilon^2 z_m^2), \quad (42)$$

where

$$\begin{aligned} \tilde{s} &\doteq \frac{z_{m+1} - z_m}{1 + \varepsilon^2 z_{m+1} z_m}, & \tilde{t} &\doteq \frac{z_m - z_{m-1}}{1 + \varepsilon^2 z_{m-1} z_m}, \\ \tilde{u} &\doteq w_{m+1} - w_m, & \tilde{v} &\doteq w_m - w_{m-1} \end{aligned}$$

Eqs. (19a, 19b) turn out to be the Bäcklund transformations of (41, 42) relating the solution (w_m, z_m) to a new solution $(\tilde{w}_m, \tilde{z}_m)$, i.e.

$$(w_m - w_{m+1})(z_m - z_{m+1}) - \alpha_2 \varepsilon^2 z_m z_{m+1} - \alpha_2 = 0, \quad (43a)$$

$$(\tilde{w}_m - \tilde{w}_{m+1})(\tilde{z}_m - \tilde{z}_{m+1}) - \alpha_2 \varepsilon^2 \tilde{z}_m \tilde{z}_{m+1} - \alpha_2 = 0, \quad (43b)$$

$$(\tilde{w}_m - \tilde{w}_{m+1})(z_m - z_{m+1}) - \alpha_2 \varepsilon^2 z_m z_{m+1} - \alpha_2 = 0. \quad (43c)$$

From (43) we evince that the solutions of (41, 42) are not independent as are related by (43a). Then given a solution (w_m, z_m) of the two equations (41, 42), solving (43c) we find \tilde{w}_m and from (43b) we find \tilde{z}_m .

A similar result one obtains in the case of (37) when $\varepsilon = 0$.

In the case of the n symmetries, if $\kappa_m = \lambda_m = 0$ the dependence on m is just parametric and thus this phenomena is no more present. We have only one non-autonomous differential difference equation and one non-autonomous Bäcklund transformation

$$\begin{aligned} w_{n,\lambda} &= \mathcal{F}_n^{(+)} B_n \left(\frac{w_{n+1} - w_{n-1}}{1 + \varepsilon^2 w_{n+1} w_{n-1}} \right) + \mathcal{F}_n^{(-)} (1 + \varepsilon^2 w_n^2) C_n (w_{n+1} - w_{n-1}), \\ (w_n - \tilde{w}_n)(w_{n+1} - \tilde{w}_{n+1}) - \varepsilon^2 \alpha_2 \left(\mathcal{F}_n^{(+)} w_{n+1} \tilde{w}_{n+1} + \mathcal{F}_n^{(-)} w_n \tilde{w}_n \right) - \alpha_2 &= 0. \end{aligned}$$

5 Conclusions.

In this article we presented results on the linearization and on the symmetries for a non-autonomous nonlinear partial difference equation belonging to the Boll classification of quad-graph equations on the lattice, the ${}_t H_1^\varepsilon$ equation.

In particular we show its explicit linearization obtained by reducing the ${}_t H_1^\varepsilon$ equation to a system of autonomous partial difference equations which can be explicitly solved and by showing that its Lax pair is fake i.e. by a gauge transformation the spectral problems turns out to be independent from the spectral parameter and the field.

We can find its three point generalized symmetries which reflect the linearizability of the discrete equation by depending on arbitrary functions of a

continuous variable and on the discrete lattice index. We plan to discuss in detail this very interesting peculiar result in a future work.

Left to future work is also the analysis of the other linearizable equations of the Boll classification.

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